A TAUBERIAN THEOREM FOR RANDOM WALK

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ABSTRACT

Let $X_1, X_2, \ldots$ be independent random variables, all with the same distribution symmetric about 0;

$$S_n = \sum_{i=1}^{n} X_i.$$

It is shown that if for some fixed interval I, constant $1 < \alpha \leqq 2$ and slowly varying function M one has

$$\sum_{k=1}^{n} P\{S_k \in I\} \sim \frac{n^{1-1/\alpha}}{M(n)} \qquad (n \to \infty),$$

then the X_i belong to the domain of attraction of a symmetric stable law.

1. **Introduction.** Let $Y_1, Y_2, \cdots$ be a Markov chain and $N_n(A) =$ number of visits to A by Y_k up till time n. A well known result of Darling and Kac ([2], especially §6) states that (under very mild conditions) $N_n(A)$ tends to infinity and has a nondegenerate limit distribution after proper normalization, only if(2)(3)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_x\{Y_k \in A\}}{n^{1-1/\alpha}\{M(n)\}^{-1}} = 1 \tag{1.1}$$

for some fixed $1 \leqq \alpha < \infty$ and slowly varying function M for which $n^{1-1/\alpha}[M(n)]^{-1} \to \infty$ $(n \to \infty)$. If (1.1) holds uniformly for $x \in A$, then $n^{-1+1/\alpha}M(n)N_n(A)$ has a Mittag-Leffler distribution as limit distribution. If Y_k, $k \geqq 1$, is a random walk, i.e. if $Y_k = S_k = \sum_{i=1}^{k} X_i$ for independent, identically distributed random variables X_i, and if A is a bounded interval, then (1.1) reduces to

Received March 10, 1968.

(1) Research supported by the National Science Foundation under grant GP 7128.

(2) $P_x[E]$ denotes the conditional probability of the event E given $Y_1 = x$.

(3) The condition in Theorem 5 of [2] is stated in terms of the generating function

$$\sum_{k=1}^{\infty} z^k P_x\{Y_k \in A\}.$$

However, by means of Karamata's Tauberian theorem this condition is easily translated into (1.1).

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P\{S_k \in A\}}{n^{1-1/\alpha}\{M(n)\}^{-1}} = 1. \tag{1.2}$$

The uniformity of the limit in (1.1) as x varies over a compact set is automatic for random walks by Corollary 1 in [8]; moreover by the estimates in §48 of [7] (see also [6]) α can take only values in the closed interval [1, 2] in the case of random walks. Professor Spitzer raised the question whether (1.2) implies that X_i belongs to the domain of attraction of a stable distribution. The purpose of this note is to prove the theorem below, which answers the question affirmatively for $1 < \alpha \leqq 2$ and symmetric X_i.

THEOREM. *Let $X_1, X_2, \cdots$ be independent random variables, all with the same distribution function $F(\cdot)$, symmetric about the origin and let $S_n = \sum_{i=1}^{n} X_i$. If for some fixed interval I*(4)*, $1<\alpha\leqq 2$ and slowly varying function M*

$$\sum_{k=1}^{n} P\{S_k \in I\} \sim \frac{n^{1-1/\alpha}}{M(n)} \qquad (n\to\infty), \tag{1.3}$$

then

$$\lim_{n\to\infty} P\left\{\frac{CS_n}{n^{1/\alpha}M(n)} \leqq x\right\} = F_\alpha(x) \tag{1.4}$$

where F_α is the symmetric stable distribution function with characteristic function $\exp -|t|^\alpha$ and C is a constant depending only on I and the support of F. If F is not a lattice distribution then(5)

$$C = \frac{\pi(\alpha-1)}{\Gamma\left(\frac{1}{\alpha}\right)|I|}.$$

REMARK. The converse implication, i.e. from (1.4) to (1.3) is a special case of Stone's local limit theorem ([9], Theorem 1). Thus (1.3) and (1.4) are actually equivalent. The local limit theorem does not require symmetry assumptions and allows $\alpha = 1$ as well. It seems likely that the present theorem will also hold in this greater generality. However, our proof makes essential use of the symmetry and of $\alpha > 1$ and therefore offers little hope for generalization.

2. **Proof of the theorem.** We shall restrict ourselves to the case where F is not a lattice distribution. For a lattice distribution the proof is almost the same and actually simpler because Lemma 1(c) is not needed. We may also exclude the

(4) More generally, by Corollary 1 of [8], we could replace I by any bounded Borel set whose boundary has zero Lebesgue measure.

(5) $|A|$ denotes the Lebesgue measure of A.

case where $\sigma^2 = \int x^2 dF(x) < \infty$ for this case is covered by the central imit theorem. $C_1, C_2, \cdots$ will denote constants (which may depend on F, α, M and I though).

First we show that F may be assumed quite smooth.

LEMMA 1. *If F is not a lattice distribution and $\sigma^2 = \infty$ and* (1.3) *holds then* (*a*) *For any fixed interval J*

$$P\{S_n \in J\} \sim \left(1 - \frac{1}{\alpha}\right) \frac{|J|}{|I|} \frac{1}{n^{1/\alpha} M(n)}. \tag{2.1}$$

(*b*) *For all sufficiently large n*

$$\sup_{\substack{h \geqq |I| \\ -\infty < u < +\infty}} \frac{1}{h} P\{S_n \in [u, u+h]\} \leqq \frac{12}{|I|} \left(1 - \frac{1}{\alpha}\right) \frac{1}{n^{1/\alpha} M(n)}. \tag{2.2}$$

(*c*) *For every $\varepsilon > 0$ and $B > 0$ there exists an $N = N(\varepsilon, B)$ such that for all $n \geqq N$*

$$\sup_{|x| \leqq B\sqrt{n}} \left| \frac{P\{S_n \in I\}}{P\{S_n \in x + I\}} - 1 \right| \leqq \varepsilon. \tag{2.3}$$

Proof. Since F is a symmetric non-lattice distribution, the smallest closed subgroup containing the support of F is the whole group of reals. By Proposition 2 and Corollary 1 of [8]

$$\lim_{n \to \infty} \frac{P\{S_{n+k} \in J\}}{P\{S_n \in I\}} = \frac{|J|}{|I|} \tag{2.4}$$

for each fixed k. In particular

$$\int_0^h ds\, P\{S_{2[n/2]} \in [-s, +s]\} \sim \frac{P\{S_n \in I\}}{|I|} \int_0^h 2s\, ds = \frac{h^2}{|I|} P\{S_n \in I\}. \tag{2.5}$$

On the other hand, if

$$\psi(t) = \int e^{itx} dF(x)$$

is the characteristic function of F, then (see [1], formula 10.3.3)

$$\int_0^h ds\, P\{S_m \in [u-s, u+s]\} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos ht}{t^2} e^{-iut} \psi^m(t)\, dt, \tag{2.6}$$

so that we conclude

$$P\{S_n \in I\} \sim \frac{|I|}{\pi h^2} \int_{-\infty}^{+\infty} \frac{1-\cos ht}{t^2} \psi^{2[n/2]}(t)\, dt. \tag{2.7}$$

Since $0 \leqq \psi^2(t) \leqq 1$ (F is symmetric) the right hand side of (2.7) is decreasing in n. Thus $P\{S_n \in I\}$ is "approximately decreasing" and it is an easy consequence of this fact and (1.3) (see [3], proof of Hilfssatz 3 in Chapter 16.1 or [4], Theorem XIII.5.5) that

$$P\{S_n \in I\} \sim \left(1 - \frac{1}{\alpha}\right) \frac{1}{n^{1/\alpha} M(n)}.$$

(2.1) now follows from Corollary 1 in [8].

As for part (b) it suffices to show

$$\sup_u P\left\{S_n \in \left[u - \frac{1}{2}|I|, u + \frac{1}{2}|I|\right]\right\} \leqq 6\left(1 - \frac{1}{\alpha}\right) \frac{1}{n^{1/\alpha} M(n)}.$$

since each interval of length $h \geqq |I|$ can be written as the union of at most $2h/|I|$ intervals of length $|I|$. But by Theorem 1 of [8] there exists a $\delta > 0$ such that for all $n \geqq n_0$ and all u

$$P\left\{S_n \in \left[u - \frac{1}{2}|I|, u + \frac{1}{2}|I|\right]\right\} \leqq \frac{3}{4} P\,\{S_{2[n/2]} \in (u - |I|, u + |I|)\} + e^{-\delta n}$$

$$\leqq \frac{3}{4|I|} \int_0^{2|I|} ds\, P\{S_{2[n/2]} \in [u - s, u + s]\} + e^{-\delta n}$$

$$= \frac{3}{4\pi|I|} \int_{-\infty}^{+\infty} \frac{1 - \cos 2|I|t}{t^2} e^{-iut} \psi^{2[n/2]}(t)\, dt + e^{-\delta n} \tag{2.8}$$

$$\leqq \frac{3}{4\pi|I|} \int_{-\infty}^{+\infty} \frac{1 - \cos 2|I|t}{t^2} \psi^{2[n/2]}(t)\, dt + e^{-\delta n}$$

$$\leqq 6\left(1 - \frac{1}{\alpha}\right) \frac{1}{n^{1/\alpha} M(n)} \qquad \text{(See (2.1) and (2.7)).}$$

This proves (b).

To prove (c) we observe that one can decompose F as

$$F = a\,G_1 + (1 - a)G_2 \tag{2.9}$$

for some symmetric distribution functions G_1, G_2 such that

$$\frac{1}{4} \leqq a \leqq \frac{3}{4} \tag{2.10}$$

and such that the support of $G_1(dx)$ is bounded. One can clearly find such functions by taking a $G_1(dx) = \alpha(x) F(dx)$ for some $0 \leqq \alpha(x) \leqq 1$, α symmetric and zero outside a compact interval and such that

$$\frac{1}{4} \leqq \int \alpha(x)\, dF(x) \leqq \frac{3}{4}.$$

$G_1(dx)$ is then obtained by normalizing $\alpha(x)$ $F(dx)$ and $(1-a)G_2(dx) = (1-\alpha(x))$ $F(dx)$. It is clear from this construction and the fact that $\sigma^2 = \infty$ that we can make

$$\sigma_1^2 = \int_{-\infty}^{+\infty} x^2\, dG_1(x)$$

as large as desired. From (2.9) we have for $n = 2m$ or $n = 2m + 1$(6)

$$F^{(n)} = \sum_{k=0}^{m} \binom{m}{k} a^k (1-a)^{m-k} G_1^{(k)} * G_2^{(m-k)} * F^{(n-m)}. \tag{2.11}$$

We shall use the abbreviation H_k for the distribution function $G_2^{(m-k)} * F^{(n-m)}$ (suppressing the dependence on n, m). We shall use $G(A)$ for the measure assigned to A by a distribution function G. Then

$$F^{(n)}(x+I) = \sum_{k=0}^{m} \binom{m}{k} a^k (1-a)^{m-k} \int H_k(dy)\, G_1^{(k)}(x-y+I). \tag{2.12}$$

Because of (2.10), there exists a $b > 0$ such that

$$\sum_{k \leqq am/2} \binom{m}{k} a^k (1-a)^{m-k} \leqq e^{-bm}, \quad m \geqq m_0. \tag{2.13}$$

Also, by Esseen's form of the central limit theorem ([5], Theorem 42.2) or by Stone's local limit theorem ([9], Theorem 1)

$$\left| G_1^{(k)}(x-y+I) - \frac{1}{\sqrt{2\pi k}\,\sigma_1} \int_I e^{-(x-y+z)^2/2k\sigma_1^2}\, dz \right| = o\left(\frac{1}{\sqrt{k}}\right) \tag{2.14}$$

uniformly in x, y. Now take C_0 such that

$$\frac{2}{\sqrt{2\pi}} \int_{C_0/2}^{\infty} e^{-u^2/2}\, du \leqq \frac{\varepsilon}{6 \cdot 24}. \tag{2.15}$$

By viture of (2.14) we can then find $m_1 = m_1(B, \varepsilon)$ such that for $k \geqq (a/2)m$, $m \geqq m_1$, $|x| \leqq B\sqrt{2m+1}$, $|y| \leqq C_0 \sigma_1 \sqrt{m}$

(6) $F^{(r)}$ is the r-fold convolution of F.

$$
\begin{aligned}
(2.16)\quad & \left| G_1^{(k)}(x-y+I) - G_1^{(k)}(-y+I) \right| \leqq \frac{\varepsilon\,|I|}{12\sqrt{2\pi k}\,\sigma_1}\, e^{-2C_0^2\sigma_1^2 m/2k\sigma_1^2} \\
& + \frac{1}{\sqrt{2\pi k}\,\sigma_1} \int_I \left| e^{-(x-y+z)^2/2k\sigma_1^2} - e^{-(-y+z)^2/2k^2\sigma_1^2} \right| dz \\
& \leqq \frac{1}{\sqrt{2\pi k}\,\sigma_1} \int_I e^{-(-y+z)^2/2k\sigma^{12}} dz \left[\frac{\varepsilon}{12} + \left| \exp\left\{ \frac{4B^2 m + 4B\sqrt{2m}\cdot C_0\sigma_1\sqrt{m}}{2k\sigma_1^2} \right\} - 1 \right| \right] \\
& \leqq 2G_1^{(k)}(-y+I) \left[\frac{\varepsilon}{12} + \left| \exp\left\{ \frac{4B^2 m + 8BC_0\sigma_1 m}{2k\sigma_1^2} \right\} - 1 \right| \right].
\end{aligned}
$$

We already pointed out that σ_1 can be taken arbitrarily large; in particular we may assume that it is so large that the factor in square brackets in the last member of (2.16) does not exceed $\varepsilon/6$ for $k \geqq am/2$. Note that the lower bound for σ_1 required here depends only on B, C_0 and a. Once σ_1 has been chosen in this way we have under the conditions for (2.16)

$$
\begin{aligned}
(2.17)\quad & \left| \int_{|y| \leqq C_0\sigma_1\sqrt{m}} H_k(dy) G_1^{(k)}(x-y+I) - \int_{|y| \leqq C_0\sigma_1\sqrt{m}} H_k(dy) G_1^{(k)}(-y+I) \right| \\
& \leqq \frac{\varepsilon}{3} \int H_k(dy) G_1^{(k)}(-y+I).
\end{aligned}
$$

To estimate the analogous integrals over $|y| > C_0\sigma_1\sqrt{m}$ we use the following inequality which is almost immediate from the definition of H_k, $n-m \geqq m$ and (2.2) (see also [7], p. 90);

$$
\begin{aligned}
\sup_{x,z} H_k(x-z+I) & \leqq \sup_u F^{(m)}([u, u+|I|]) \\
& \leqq 12\left(1-\frac{1}{\alpha}\right) \frac{1}{m^{1/\alpha} M(m)}, \qquad m \geqq m_2 .
\end{aligned}
$$

This inequality implies for all $|u| \leqq B\sqrt{2m+1}$, $m \geqq m_2$,

$$
\begin{aligned}
& \int_{|y| \geqq C_0\sigma_1\sqrt{m}} H_k(dy) G_1^{(k)}(u-y+I) \\
(2.18)\quad & = \iint_{\substack{z+y \in u+I \\ |y| \geqq C_0\sigma_1\sqrt{m}}} H_k(dy) G_1^{(k)}(dz) \leqq \int_{|z| \geqq (C_0\sigma_1 - 2B)\sqrt{m}} G_1^{(k)}(dz) H_k(u-z+I) \\
& \leqq 12\left(1-\frac{1}{\alpha}\right) \frac{1}{m^{1/\alpha} M(m)} \int_{|z| \geqq (C_0\sigma_1 - 2B)\sqrt{m}} G_1^{(k)}(dx).
\end{aligned}
$$

Without loss of generality we may assume σ_1 so large that $2B \leqq \frac{1}{2}C_0\sigma_1$ and then for $m \geqq m_3$, $am/2 \leqq k \leqq m$

$$\text{(21.9)}\quad \int_{|z| \geqq (C_0\sigma_1 - 2B)\sqrt{m}} G_1^{(k)}(dz) \leqq \int_{|z| \geqq C_0\sigma_1\sqrt{k}/2} G_1^{(k)}(dz) \leqq \frac{2}{\sqrt{2\pi}} \int_{|u| \geqq C_0/2} e^{-u^2/2}du \leqq \frac{\varepsilon}{6.24}$$

(for the last two steps we used the central limit theorem and (2.15)). We now combine (2.12), (2.13), (2.17)–(2.19) to obtain for $n = 2m$ or $2m+1$, $|x| \leqq B\sqrt{2m+1}$, $m \geqq \max(m_0, m_1, m_2, m_3)$

$$\text{(2.20)}\quad \begin{aligned} &|F^{(n)}(x+I) - F^{(n)}(I)| \\ &\leqq 2e^{-bm} + \sum_{am/2 \leqq k \leqq m} \binom{m}{k} a^k(1-a)^{m-k} \left[\frac{\varepsilon}{3} \int H_k(dy)G_1^{(k)}(-y+I) + \frac{\varepsilon}{6}\left(1 - \frac{1}{\alpha}\right)\frac{1}{m^{1/\alpha}M(m)}\right] \\ &\leqq 2e^{-bm} + \frac{\varepsilon}{3}F^{(n)}(I) + \frac{\varepsilon}{3}\left(1 - \frac{1}{\alpha}\right)\frac{1}{n^{1/\alpha}M(n)}. \end{aligned}$$

In view of (2.1) this implies (2.3) for n large enough (recall $F^{(n)}(I) = P\{S_n \in I\}$) and the proof of Lemma 1 is therefore complete.

Practically the only reason for proving Lemma 1 is that it allows us to replace F by $F * \Phi$ where Φ is the standard normal distribution with density $1/\sqrt{2\pi}e^{-x^2/2}$. Indeed, let $Y_1, Y_2, \cdots$ be independent normal variables, each with distribution Φ and assume that the $\{Y_i\}_{i \geqq 1}$ are independent of the $\{X_i\}_{i \geqq 1}$. Then, by Lemma 1, there exists an $N_1(\varepsilon, B)$ such that for $n \geqq N_1(\varepsilon, B)$

$$\text{(2.21)}\quad \begin{aligned} &\left|P\{S_n + \sum_{i=1}^{n} Y_i \in I\} - P\{S_n \in I\}\right| \\ &\leqq \int \Phi^{(n)}(dy)\,|P\{S_n + y \in I\} - P\{S_n \in I\}| \\ &\leqq \int_{|y| \leqq B\sqrt{n}} \Phi^{(n)}(dy)\varepsilon P\{S_n \in I\} + \int_{|y| > B\sqrt{n}} \Phi^{(n)}(dy)24\left(1 - \frac{1}{\alpha}\right)\frac{1}{n^{1/\alpha}M(n)} \\ &\leqq \left(1 - \frac{1}{\alpha}\right)\frac{1}{n^{1/\alpha}M(n)}\left(2\varepsilon + \frac{24}{\sqrt{2\pi}} \int_{|u| \geqq B} e^{-u^2/2}du\right). \end{aligned}$$

Since B can be taken arbitrarily large and ε arbitraily small we see from (2.1) and (2.21) that

$$P\left\{S_n + \sum_{i=1}^{n} Y_i \in I\right\} \sim \left(1 - \frac{1}{\alpha}\right)\frac{1}{n^{1/\alpha}M(n)} \qquad (n \to \infty)$$

as well. In other words the random variables $X_i + Y_i$ satisfy the hypothesis of the theorem. In addition

$$\lim_{n\to\infty} P\left\{\frac{C\left(S_n + \sum_{i=1}^{n} Y_i\right)}{n^{1/\alpha}M(n)} \leqq x\right\} = F_\alpha(x)$$

is equivalent to (1.4) because

$$\frac{\sum_{i=1}^{n} Y_i}{n^{1/\alpha}M(n)} \to 0 \qquad \text{in probability}$$

(this is even true for $\alpha = 2$, for then $\sigma^2 = \infty$ implies that $M(n)$ must tend to ∞ if (1.3) is to hold).(7) We therefore see that it suffices to prove the theorem for $X_i + Y_i$ instead of X_i. Rather than carry the Y_i along we change notation and write just X_i for $X_i + Y_i$ and F again for the distribution function of the new F. *In the sequel we therefore have*

$$(2.22) \qquad |\psi(t)| = \left|\int e^{itx}dF(x)\right| \leqq \left|\int e^{itx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\right| = e^{-t^2/2}.$$

By the standard inversion formula, [5] Theorem 12.1, (2.1) now gives for any $\varepsilon > 0$

$$P\left\{S_n \in \left[-\frac{1}{2}, +\frac{1}{2}\right]\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sin\frac{t}{2}}{t}\psi^n(t)\,dt$$
$$= \frac{1}{2\pi}\int_{-\varepsilon}^{+\varepsilon} \frac{2\sin\frac{t}{2}}{t}\psi^n(t)dt + 0\left(\int_{\varepsilon}^{\infty} e^{-nt^2/2}dt\right).$$

Since this holds for each ε and $\psi(t) \geqq 0$ for $|t|$ sufficiently small, we can translate the basic hypothesis of our theorem into

$$(2.23) \qquad \frac{1}{2\pi}\int_{-\varepsilon}^{+\varepsilon}\psi^n(t)dt \sim \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi^n(t)\,dt \sim \left(1-\frac{1}{\alpha}\right)\frac{1}{|I|}\frac{1}{n^{1/\alpha}M(n)}$$
$$(n \to \infty,\ \varepsilon > 0).$$

Next we show (and this is the crux of the proof) that

(7) See for instance Theorem 4.1 in C. G. Esseen, On the concentration-function of a sum of independent random variables, Z. Wahrscheinlichkeitstheorie verw. Gebiete 9 (1968) 290–308.

$$A_n = \frac{1}{\left(1 - \frac{1}{\alpha}\right)} |I| n^{1/\alpha} M(n)$$

is of the right order to normalize S_n. For this purpose we define

$$d_n(q) = \inf\{L: P\{S_n \in [-L, +L]\} \geqq q\}, \qquad 0 \leqq q < 1.$$

It follows directly from Theorem 3 in [6] and Lemma 1 above that for any $0 < q_1 \leqq q_2 < 1$ there exists a $C_1(q_1, q_2)$ such that

$$\limsup_{n\to\infty} \frac{d_n(q_2)}{d_n(q_1)} \leqq C_1(q_1, q_2) < \infty. \tag{2.24}$$

(We should point out that $d_n(q)$ is not exactly the same as the dispersion function $D(S_n; q)$ used in [6]. But clearly $D(S_n; q) \leqq 2d_n(q)$ whereas for $q > \frac{1}{2}$, $d_n(q) \leqq D(S_n; q)$ since any interval containing S_n with probability $q > \frac{1}{2}$ must contain the origin for symmetrically distributed S_n.) Now it is clear from (2.2) that for large n

$$d_n(q) \geqq \frac{q}{24} A_n, \tag{2.25}$$

so that only an upper bound for d_n is needed.

LEMMA 2. *For each $q < 1$ there exists a $C_2(q)$ such that*

$$d_n(q) \leqq C_2(q) A_n. \tag{2.26}$$

Proof. Since

$$\int P\{S_n \in dx\} \int_{-\varepsilon}^{+\varepsilon} \psi(t) \frac{1 \cos xt}{1 - \psi(t)} dt = \int_{-\varepsilon}^{+\varepsilon} \psi(t) \frac{1 - \psi^n(t)}{1 - \psi(t)} dt$$

$$= \sum_{k=1}^{n} \int_{-\varepsilon}^{+\varepsilon} \psi^k(t)\, dt \sim \frac{2\pi}{|I|} n^{1-1/\alpha} \{M(n)\}^{-1}$$

we have for sufficiently large n (from the positivity of the integrand)

$$P\left\{\int_{-\varepsilon}^{+\varepsilon} \psi(t) \frac{1 - \cos t S_n}{1 - \psi(t)} dt \geqq \frac{10\pi}{|I|} n^{1-1/\alpha} (M(n))^{-1}\right\} \leqq \frac{1}{4}.$$

In particular (because $P\{d_n(\frac{1}{4}) \leqq S_n \leqq d_n(\frac{3}{4})\} \geqq \frac{1}{2}$ and $\psi(t) \geqq \frac{1}{2}$ for $|t| \leqq \pi\{d_n(\frac{1}{4})\}^{-1}$ eventually), for $n \geqq n_1$ we can find an

$$x_n \in \left[d_n\left(\frac{1}{4}\right), d_n\left(\frac{3}{4}\right)\right] \tag{2.27}$$

for which

$$\frac{1}{\pi^2}\int_{|t|\leqq \pi/x_n}\frac{t^2x_n^2}{1-\psi(t)}dt \leqq \int_{-\varepsilon}^{+\varepsilon}\psi(t)\,\frac{1-\cos tx_n}{1-\psi(t)}\,dt\,. \tag{2.28}$$
$$\leqq \frac{10\pi}{|I|}\; n^{1-1/\alpha}\{M(n)\}^{-1}.$$

To compare n with x_n we now define the function $K(\cdot)$ by

$$y^\alpha K(y) = \inf\{k : k^{1/\alpha}M(k) \geqq y\} \qquad y \geqq 0.$$

From Karamata's well known representation for slowly varying functions, [4], Corollary to Theorem VIII.9.1, one easily sees that for suitable $n_2 = n_2(\varepsilon)$

$$\inf_{m \geqq (1+\varepsilon)n \geqq n_2} \frac{m^{1/\alpha}M(m)}{n^{1/\alpha}M(n)} \geqq \left(1+\frac{\varepsilon}{2}\right)^{1/\alpha}. \tag{2.29}$$

From this property it follows immediately that K is also a slowly varying function and that

$$K(n^{1/\alpha}M(n)) \sim M^{-\alpha}(n) \qquad (n \to \infty) \tag{2.30}$$

as well as

$$M(y^\alpha K(y)) \sim K^{-1/\alpha}(y) \qquad (y \to \infty). \tag{2.31}$$

We now observe that by (2.27) and (2.25)

$$x_n \geqq d_n\left(\frac{1}{4}\right) \geqq \frac{|I|}{96}\,\frac{1}{\left(1-\frac{1}{\alpha}\right)}\,n^{1/\alpha}M(n).$$

By the definition of K and (2.29) this implies for $n \geqq n_3$

$$n \leqq \left\{\frac{100}{|I|}x_n\left(1-\frac{1}{\alpha}\right)\right\}^\alpha K\left(\frac{100}{|I|}x_n\left(1-\frac{1}{\alpha}\right)\right). \tag{2.32}$$

In view of the slowly varying character of M and K and the analogue of (2.29) obtained by replacing $1/\alpha$ by $1-1/\alpha$ and M by M^{-1}, (2.32) implies

$$n^{1-1/\alpha}\{M(n)\}^{-1} \;\leqq\; C_3x_n^{\alpha-1}K(x_n)^{1-1/\alpha}\{M(x_n^\alpha K(x_n))\}^{-1}$$
$$\sim\; C_3x_n^{\alpha-1}K(x_n) \qquad \text{(See (2.31))}.$$

This estimate of the last member of (2.28) leads to

$$\int_{\pi/C_1^3x_n \leqq |t| \leqq \pi/x_n}\frac{dt}{1-\psi(t)} \leqq C_1^6C_4x_n^{\alpha-1}K(x_n), \qquad n \geqq n_4. \tag{2.33}$$

C_1 in (2.33) is taken as $C_1(\frac{1}{4},\frac{3}{4})$, which we assume > 1 without loss of generality, whereas C_3 and C_4 are constants depending on α and $|I|$ only. Next observe that

(2.34) $$d_{n+1}\left(\frac{1}{4}\right) \leqq d_n\left(\frac{3}{4}\right) \quad \text{for } n \geqq n_5,$$

because for each C_5

(2.35) $$P\left\{|S_{n+1}| \leqq d_n\left(\frac{3}{4}\right)\right\} \geqq P\left\{|S_n| \leqq d_n\left(\frac{3}{4}\right) - C_5\right\} P\{|X_{n+1}| \leqq C_5\};$$

in particular if C_5 is fixed so large that $P\{|X_{n+1}| \leqq C_5\} \geqq \frac{1}{2}$, then it follows from

$$\liminf_{n\to\infty} P\left\{|S_n| \leqq d_n\left(\frac{3}{4}\right) - C_5\right\}$$

$$= \liminf_{n\to\infty} P\left\{|S_n| \leqq d_n\left(\frac{3}{4}\right)\right\} \quad (\text{see } (2.2)) \geqq \frac{3}{4}$$

and (2.35) that eventually

$$P\left\{|S_{n+1}| \leqq d_n\left(\frac{3}{4}\right)\right\} > \frac{1}{4},$$

whence (2.34). (2.27) together with (2.34) and (2.24) shows

$$x_{n+1} \leqq \frac{d_n\left(\frac{3}{4}\right)}{d_n\left(\frac{1}{4}\right)} \frac{d_{n+1}\left(\frac{3}{4}\right)}{d_{n+1}\left(\frac{1}{4}\right)} x_n \leqq C_1^2\left(\frac{1}{4}, \frac{3}{4}\right) x_n,$$

so that each interval of the form $[1/\pi C_1^k, 1/\pi C_1^{k-2}]$, $k \geqq k_1$ contains at least one x_n^{-1}. This finally allows us to convert (2.33) into

$$\int_{C_1^{-k-1} \leqq |t| \leqq C_1^{-k}} \frac{dt}{1-\psi(t)} \leqq C_1^6 C_6 C_1^{k(\alpha-1)} K(C_1^k),\ k \geqq k_1,$$

an estimate which is free of x_n. It follows immediately that for $k \geqq k_2$

$$\int_{C_1^{-k-1} \leqq |t| \leqq C_1^{-k}} |\psi^n(t)|\, dt \leqq \int_{C_1^{-k-1} \leqq |t| \leqq C_1^{-k}} e^{-n(1-\psi(t))} \frac{1-\psi(t)}{1-\psi(t)} dt$$

$$\leqq \frac{1}{n} \max_{x \geqq 0} x e^{-x} \int_{C_1^{-k-1} \leqq |t| \leqq C_1^{-k}} \frac{dt}{1-\psi(t)} \leqq \frac{C_7}{n} C_1^{k(\alpha-1)} K(C_1^k).$$

$k_2 \geqq k_1$ only has to be chosen such that $\psi(t) \geqq 0$ for $|t| \leqq C_1^{-k_2}$. If we write C_8 for $C_1^{-k_2}$ we arrive at

$$\begin{aligned}(2.36)\qquad \int\limits_{C/A_n \leqq |t| \leqq C_8} |\psi^n(t)|\,dt &\leqq \frac{C_7}{n} \sum_{C_1{}^k \leqq A_n/C} C_1^{k(\alpha-1)} K(C_1^k) \\ &\leqq \frac{C_1}{n}\left(\frac{A_n}{C}\right)^{\alpha-1} K(A_n) \leqq \frac{C_{10}}{A_n}\,\frac{1}{C^{\alpha-1}}, \qquad n \geqq n_6\end{aligned}$$

(see the definition of A_n and (2.30)); C_{10} is independent of C.

The proof of the lemma is now completed by an application of the inversion formula, [5] Theorem 12.1, which gives

$$P\left\{|S_n| \leqq \frac{\pi}{2}\,\frac{A_n}{C}\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sin\frac{\pi}{2}A_nC^{-1}t}{t}\,\psi^n(t)\,dt$$

$$\begin{aligned}(2.37)\qquad &\geqq \frac{2}{2\pi}\,\frac{A_n}{C}\int\limits_{|t| \leqq C/A_n} \psi^n(t)\,dt - \frac{1}{2}\,\frac{A_n}{C}\int\limits_{|t| > C/A_n} |\psi^n(t)|\,dt \\ &\geqq \frac{A_n}{\pi C}\int_{-\infty}^{+\infty} \psi^n(t)\,dt - \frac{A_n}{C}\int\limits_{C/A_n < |t| \leqq C_8} |\psi^n(t)|\,dt \\ &- \frac{A_n}{C}\int\limits_{|t| > C_8} e^{-nt^2/2}dt \geqq \frac{2}{C} - \frac{2C_{10}}{C^\alpha} \qquad \text{(See (2.23) and (2.36)).}\end{aligned}$$

Since $\alpha > 1$ we can choose $C = C_{11} > 0$ such that the last member of (2.37) exceeds C_{11}^{-1} so that $d(C_{11}^{-1}) \leqq \pi A_n/2C_{11}$. This proves (2.26) for $q = C_{11}^{-1}$ and for general q it then follows from (2.24) and the monotonicity of $d_n(\cdot)$.

To prove the theorem is easy enough now. By (2.25) and (2.26) every increasing sequence of integers contains a subsequence along which the distribution of $D\,S_n/A_n$ converges weakly to a nondegenerate distribution (D a positive constant). Consider then any sequence $n_i \to \infty$ such that the weak limit

$$\lim_{i\to\infty} P\left\{\frac{D\,S_{n_i}}{A_{n_i}} \leqq x\right\} = G(x)$$

exists. Then also

$$\lim_{i\to\infty} \psi^{n_i}\left(\frac{Dt}{A_{n_i}}\right) = \gamma(t) = \int e^{itx}\,dG(x)$$

and it suffices to prove $\gamma(t) = \exp - |t|^\alpha$ when D is properly chosen(8). To begin

(8) If ε is taken such that $\psi(t) \geqq 0$ for $|t| \leqq \varepsilon$, then one easily deduces from (2.23) and Karamata's Tauberian theorem that

$$|\{t: |t| \leqq \varepsilon \text{ and } \psi(t) \geqq y\}| \sim 2C(1-y)^{1/\alpha}M^{-1}(1/1-y) \text{ as } y \uparrow 1\left(C = \frac{\pi(\alpha-1)}{\Gamma(1/\alpha)|I|}\right)$$

(compare also (2.43) and (2.44) below). One would like to conclude from this that

$$1 - \psi(t) \sim \frac{t^\alpha}{C^\alpha K(1/t)} \text{ as } t \downarrow 0,$$

which is equivalent to the main result (1.4). The author did not succeed in constructing a rigorous proof along these simple lines.

with we have the following simple estimate because X_i has a symmetric distribution ([4], Lemma V.5.2):

$$\begin{aligned}\frac{1}{2}P\left\{\max_{1\leqq i\leqq n}|X_i| \geqq 2C_2\left(\frac{3}{4}\right)A_n\right\} &\leqq P\left\{|S_n| \geqq 2C_2\left(\frac{3}{4}\right)A_n\right\}\\ &\leqq P\left\{|S_n| \geqq 2d_n\left(\frac{3}{4}\right)\right\} \leqq \frac{1}{4}.\end{aligned} \tag{2.38}$$

In turn, (2.38) implies

$$nP\left\{|X_1| \geqq 2C_2\left(\frac{3}{4}\right)A_n\right\} \leqq C_{12}$$

or, using the definition of A_n and $K(y)$,

$$P\{|X_1| \geqq y\} \leqq \frac{C_{13}}{y^\alpha K(y)}, \qquad y \geqq y_1. \tag{2.39}$$

For $\alpha = 2$ this estimate will not be sharp enough, but for $\alpha < 2$ we obtain

$$\begin{aligned}1-\psi(t) &= \int_{-\infty}^{+\infty}(1-e^{itx})\,dF(x) = -\int_0^\infty (1-\cos tx)\,dP\{|X_1|\geqq x\}\\ &\leqq 2P\{|X_1| \geqq |t|^{-1}\} + \int_0^{|t|^{-1}} t^2 x P\{|X_1| \geqq x\}\,dx \leqq C_{14}\frac{|t|^\alpha}{K\left(\frac{1}{|t|}\right)}.\end{aligned} \tag{2.40}$$

As a first estimate for γ we therefore have

$$\begin{aligned}1-\gamma(t) &\leqq 1 - \liminf_{n\to\infty}\psi^n\left(\frac{Dt}{A_n}\right) \leqq 1 - \liminf_{n\to\infty}\left[1 - \frac{C_{14}D^\alpha|t|^\alpha}{A_n^\alpha K\left(\frac{A_n}{|t|}\right)}\right]^n\\ &\leqq 1 - \liminf_{n\to\infty}\left(1 - C_{15}\frac{D^\alpha}{n}|t|^\alpha\right)^n = 1-\exp\{-D^\alpha C_{15}|t|^\alpha\}.\end{aligned} \tag{2.41}$$

By means of standard estimates (e.g. [5], §13) for the tail of G in terms of the behavior of its characteristic function γ near the origin it is seen from (2.41) that

$$1 - G(x) + G(-x) = 0(x^{-\alpha}) \qquad (x\to\infty)$$

and since $\alpha > 1$ this implies that $\int|x|\,dG(x)$ is finite and that $\gamma(\cdot)$ is continuously differentiable. We also see from (2.41) that $\gamma(t) > 0$ for all t.

Much more precise information about γ is obtained by computing for any integer $k \geqq 1$

$$\int_{-\infty}^{+\infty} \gamma^k(t)\,dt = \lim_{T\to\infty} \int_{-T}^{+T} \lim_{i\to\infty} \psi^{kn_i}\left(\frac{Dt}{A_{n_i}}\right) dt$$

$$= \lim_{T\to\infty}\lim_{i\to\infty} \frac{A_{n_i}}{D} \int_{|s|\le DTA_{n_i}^{-1}} \psi^{kn_i}(s)\,ds$$

$$= \lim_{i\to\infty} \frac{A_{n_i}}{D} \int_{-\infty}^{+\infty} \psi^{kn_i}(s)\,ds + \lim_{T\to\infty}\lim_{i\to\infty} O\left(\frac{A_{n_i}}{A_{kn_i}}\frac{1}{T^{\alpha-1}} + A_{n_i}\int_{C_8}^{\infty} e^{-kn_i s^2/2} ds\right)$$

(see (2.36) and (2.22)) $= 2\pi/Dk^{1/\alpha}$ (see (2.23)).
We now choose $D = \alpha\pi/\Gamma(1/\alpha)$, which is indeed equivalent to taking

$$C = \frac{\pi(\alpha-1)}{\Gamma\left(\frac{1}{\alpha}\right)|I|}$$

in (1.4). For this choice of D

$$\int_{-\infty}^{+\infty} \gamma^k(t)\,dt = \frac{2\Gamma\left(\frac{1}{\alpha}\right)}{\alpha k^{1/\alpha}} = \int_{-\infty}^{+\infty} e^{-k|t|^\alpha}\,dt. \tag{2.42}$$

Introduce

$$v(y) = |\{t: \gamma(t) \geqq y\}|, \qquad 0 \leqq y \leqq 1.$$

Because $0 \leqq \gamma(t) \leqq 1$ the left hand side of (2.42) can then be rewritten as

$$\int_{-\infty}^{+\infty} \gamma^k(t)\,dt = -\int_0^1 x^k dv(x), \qquad k = 1,2,\cdots. \tag{2.43}$$

Since the finite measure $-x\,dv(x)$ is uniquely determined by its moments (see [4], Chapter VII.3 or use Theorem 2.9.3 in [3] after an integration by parts), (2.42) and (2.43) imply

$$v(y) = |\{t: e^{-|t|^\alpha} \geqq y\}| = 2\left(\log\frac{1}{y}\right)^{1/\alpha}. \tag{2.44}$$

To complete the proof (for $\alpha < 2$) we show that $\gamma(t)$ is strictly decreasing on $t \geqq 0$. Indeed, if there exist $0 \leqq t_1 < t_2$ with $\gamma(t_1) \leqq \gamma(t_2)$ then $\min_{0\leqq t\leqq t_2}\gamma(t)$ is taken on at a point $t_3 \in (0, t_2)$ (because also $\gamma(0) = 1 > \min_{0\leqq t\leqq t_2}\gamma(t)$ for any $t_2 > 0$ since G is non degenerate by (2.25); see [5], Theorem 14.2). At t_3 we must have $\gamma'(t_3) = 0$ and $0 < z = \gamma(t_3) < 1$ (recall that $\gamma(t) > 0$ for all t). But this is impossible for then

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}|\{t: z-\varepsilon \leqq \gamma(t) \leqq z+\varepsilon\}| = \infty$$

whereas this limit should have the finite value

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[v(z-\varepsilon) - v(z+\varepsilon)\right] = \frac{4}{\alpha z}\left(\log \frac{1}{z}\right)^{1/\alpha - 1}.$$

Thus $\gamma(t)$ is strictly decreasing on $t \geqq 0$ and because γ is symmetric $v(y) = 2t(y)$ where $t(y)$ is the unique $t \geqq 0$ with $\gamma(t) = y$. This means $t(y) = (\log 1/y)^{1/\alpha}$, $\gamma((\log 1/y)^{1/\alpha}) = y$ or $\gamma(t) = \exp - |t|^{\alpha}$ as desired.

For $\alpha < 2$ the proof is complete, but for $\alpha = 2$ an extra argument is needed because the last estimate in (2.40) could conceivably fail. We show that (2.40) is correct even for $\alpha = 2$. After (2.40) the proof did not rely on $\alpha < 2$ and can therefore be used also if $\alpha = 2$. To obtain (2.40) for $\alpha = 2$ we put

$$\sigma^2(T) = \int_{-T}^{+T} x^2 \, dF(x).$$

Clearly $\sigma^2(\cdot)$ is nondecreasing and

$$1 - \psi(t) = \int (1 - \cos xt) \, dF(x) \geqq \frac{2}{\pi^2} \int_{|x| \leqq \pi/t} t^2 x^2 dF(x) = \frac{2t^2}{\pi^2} \sigma^2\left(\frac{\pi}{t}\right).$$

Therefore

$$(2.45) \qquad \int_{|t| \leqq C/A_n} \psi^n(t) \, dt \leqq \int_{|t| \leqq C/A_n} \exp\{-n(1 - \psi(t))\} \, dt$$

$$\leqq \int_{-\infty}^{\infty} \exp\left\{-\frac{2nt^2}{\pi^2} \sigma^2\left(\frac{\pi A_n}{C}\right)\right\} dt = \frac{\pi^{3/2}}{\sqrt{2n}} \frac{1}{\sigma\left(\frac{\pi A_n}{C}\right)}$$

Together with (2.23), (2.36) and (2.22), (2.45) implies

$$\frac{\pi^{3/2}}{\sqrt{2n}} \frac{1}{\sigma\left(\frac{\pi A_n}{C}\right)} + \frac{C_{10}}{A_n} \frac{1}{C^{\alpha-1}} \geqq \frac{\pi}{A_n}, \qquad n \geqq n_7.$$

Thus for $C = C_{16}$ say, C_{16} sufficiently large,

$$\sigma\left(\frac{\pi A_n}{C_{16}}\right) \leqq C_{17} \frac{A_n}{\sqrt{n}} = C_{18} M(n) \qquad (\alpha = 2),$$

or, in view of the definition of A_n and (2.31)

$$\sigma(y) \leqq C_{19} K^{-1/2}(y).$$

Thus, by virtue of (2.39),

$$1-\psi(t)=\int_{-\infty}^{+\infty}(1-\cos tx)\,dF(x)\leqq 2\,P\left\{|X_1|\geqq\frac{1}{|t|}\right\}+\frac{1}{2}t^2\sigma^2\left(\frac{1}{|t|}\right)$$

$$\leqq\frac{2C_{13}t^2}{K\left(\frac{1}{|t|}\right)}+\frac{1}{2}t^2\frac{C_{19}}{K\left(\frac{1}{|t|}\right)},$$

which is the desired replacement for (2.40).

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